

# Entangled state representations in noncommutative quantum mechanics \*

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## Abstract

We introduce new representations to formulate quantum mechanics on noncommutative coordinate space, which explicitly display entanglement properties between degrees of freedom of different coordinate components and hence could be called entangled state representations. Furthermore, we derive unitary transformations between the new representations and the ordinary one used in noncommutative quantum mechanics (NCQM) and obtain eigenfunctions of some basic operators in these representations. To show the potential applications of the entangled state representations, a two-dimensional harmonic oscillator on the noncommutative plane with both coordinate-coordinate and momentum-momentum couplings is exactly solved.

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## 1 Introduction

As is well known, representations and transformation theories, founded by Dirac [1], play basic and important role in quantum mechanics. Many quantum mechanics problems were solved cleverly by working in specific representations. Some representations, such as, the coordinate, the momentum, the number representation, as well as the coherent state representation, are often employed in the literature of ordinary quantum mechanics. In noncommutative quantum mechanics (NCQM) [2], because of the non-commutativity of coordinate-component operators, there are no common eigenstates for these different coordinate operators, and one can hardly construct a coordinate representation in the usual sense. However, in order to formulate quantum mechanics on a noncommutative space so that some dynamic problems can be solved, we do need some appropriate representations. On the other hand, we realize that in NCQM ordinary products are usually replaced by  $\ast$ -products between functions on the noncommutative space [3], which is equivalent to

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work in some kind of "quasi-coordinate" representation (in this representation, the state vectors, for example,  $|x, y\rangle$ , are not common eigenstates of the coordinate operators  $\hat{X}$  and  $\hat{Y}$  in NCQM. For details, please see Sec. 4). Of course, if we have more practical representations for NCQM, it will be more powerful to deal with the dynamic problems in NCQM.

Noticing that although two coordinate-component operators on the noncommutative space do not commute each other, the difference of the two coordinate operators indeed commute with the sum of the relevant two momentum operators, thus we can still employ Einstein-Podolsky-Rosen's (EPR) [4] idea to construct entangled states on the noncommutative space. It is easily to show that the entangled states with continuum variables are orthonormal and satisfy completeness relations, therefore they present new representations for NCQM. The first bipartite entangled state representation of continuum variables is constructed by one of the authors (H. Fan) and J. R. Klauder [5] based on the idea of quantum entanglement initiated by EPR who used commutative property of two particles' relative coordinate and total momentum. Here we extend the formalism in [5] to NCQM and investigate some basic properties of the entangled state representations on the noncommutative space. We also derive explicit unitary operators which connect the entangled state representations and the "quasi-coordinate" representation and transfer them each other. Using the unitary transformation, it is convenient to obtain eigenfunctions of some basic operators of NCQM in one representation if one knows them in another representation. To show the potential applications of the entangled state representations in NCQM, we solve exactly the energy level of a two-dimensional harmonic oscillator on a noncommutative plane with both kinetic coupling and elastic coupling.

The work is arranged as follows: In Sec. 2 we construct the entangled state representations for NCQM and derive matrix elements of coordinate and momentum operators in these representations. In order to demonstrate these states are indeed the entangled states, we study their Schmidt decompositions in Sec. 3. In Sec. 4 we investigate the transformation between the "quasi-coordinate" and the entangled state representations, and derive an explicit unitary operator which transforms them each other. Sec. 5 is devoted to study eigenfunctions of some basic operators on the noncommutative plane in the entangled state representation, which all display some extent of entanglement between the coordinates and the momenta. In Sec. 6 we study a two-dimensional harmonic oscillator on a noncommutative plane with both kinetic coupling and elastic coupling and solve its energy spectrum exactly. Some summary and discussion are presented in Sec. 7.

## 2 Entangled state representations for NCQM

Without loss of generality and for the sake of simplicity, we only discuss the noncommutative plane case in the follows. Operators  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{P}_x$  and  $\hat{P}_y$  satisfy the following commutation relations

$$[\hat{X}, \hat{Y}] = i\theta, \quad [\hat{X}, \hat{P}_x] = i, \quad [\hat{Y}, \hat{P}_y] = i, \quad (1)$$

and other commutators of these operators are vanishing, where  $\theta$  is a real parameter reflecting the non-commutativity of space coordinates, and we take  $\hbar = 1$ . Considering the following operators

$$\hat{R} = \frac{\hat{X} - \hat{Y}}{\sqrt{2}}, \quad \hat{P} = \frac{\hat{P}_x + \hat{P}_y}{\sqrt{2}}, \quad \hat{S} = \frac{\hat{X} + \hat{Y}}{\sqrt{2}}, \quad \hat{K} = \frac{\hat{P}_x - \hat{P}_y}{\sqrt{2}}. \quad (2)$$

Obviously  $\hat{R}$  and  $\hat{P}$  are commute each other, as well as  $\hat{S}$  and  $\hat{K}$  are commute each other, respectively. Thus  $\hat{R}$  and  $\hat{P}$  have common eigenstates  $|\eta\rangle$ , and  $\hat{S}$  and  $\hat{K}$  have common eigenstates  $|\xi\rangle$ . Here  $\eta$  and  $\xi$  may be complex numbers, ( $\eta = \eta_1 + i\eta_2$ , and  $\xi = \xi_1 + i\xi_2$ ), and  $\eta_1, \eta_2, \xi_1$  and  $\xi_2$  are real numbers.

In order to get explicit expressions of the eigenstates  $|\eta\rangle$  and  $|\xi\rangle$ , we use the following transformations

$$\hat{X} = x - \frac{\theta}{2} p_y, \quad \hat{Y} = y + \frac{\theta}{2} p_x, \quad \hat{P}_x = p_x, \quad \hat{P}_y = p_y, \quad (3)$$

where the operators  $x, y, p_x$  and  $p_y$  satisfy ordinary Heisenberg commutation relations

$$[x, p_x] = i, \quad [y, p_y] = i, \quad (4)$$

and other commutators of these operators are vanishing. Furthermore, introducing two independent ordinary bosonic creation and annihilation operators  $a^\dagger, a$  and  $b^\dagger, b$  with commutation relations  $[a, a^\dagger] = 1, [b, b^\dagger] = 1$ , we have

$$x = \frac{a + a^\dagger}{\sqrt{2}}, \quad p_x = \frac{a - a^\dagger}{\sqrt{2}i}, \quad y = \frac{b + b^\dagger}{\sqrt{2}}, \quad p_y = \frac{b - b^\dagger}{\sqrt{2}i}. \quad (5)$$

In terms of these creation and annihilation operators, we can express the operators  $\hat{X}, \hat{Y}, \hat{P}_x$  and  $\hat{P}_y$  as

$$\begin{aligned} \hat{X} &= \frac{a + a^\dagger}{\sqrt{2}} - \frac{\theta(b - b^\dagger)}{2\sqrt{2}i}, & \hat{P}_x &= \frac{a - a^\dagger}{\sqrt{2}i}, \\ \hat{Y} &= \frac{b + b^\dagger}{\sqrt{2}} + \frac{\theta(a - a^\dagger)}{2\sqrt{2}i}, & \hat{P}_y &= \frac{b - b^\dagger}{\sqrt{2}i}. \end{aligned} \quad (6)$$

Thus the operators  $\hat{R}$  and  $\hat{P}$  may be expressed as

$$\hat{R} = \frac{1}{2} (a + a^\dagger - b - b^\dagger) - \frac{\theta}{4i} (a - a^\dagger + b - b^\dagger), \quad \hat{P} = \frac{1}{2i} (a - a^\dagger + b - b^\dagger). \quad (7)$$

The common eigenstate  $|\eta\rangle$  of  $\hat{R}$  and  $\hat{P}$  can be written as

$$|\eta\rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|\eta|^2}{2} + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger\right) |00\rangle, \quad (8)$$

where  $|00\rangle$  is a two-mode bosonic vacuum state satisfying  $a|00\rangle = 0$  and  $b|00\rangle = 0$ . It is easily to see that

$$\frac{1}{2} (a + a^\dagger - b - b^\dagger) |\eta\rangle = \eta_1 |\eta\rangle, \quad \frac{1}{2i} (a - a^\dagger + b - b^\dagger) |\eta\rangle = \eta_2 |\eta\rangle, \quad (9)$$

which lead to

$$\hat{R}|\eta\rangle = \left(\eta_1 - \frac{\theta}{2}\eta_2\right)|\eta\rangle, \quad \hat{P}|\eta\rangle = \eta_2|\eta\rangle. \quad (10)$$

Here we would like to give an explicitly proof of the completeness relation for the eigenstates  $|\eta\rangle$  using a method of Integration within Ordered Product (IWOP) of products [6]

$$\begin{aligned} \int_{-\infty}^{\infty} d^2\eta |\eta\rangle\langle\eta| &= \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} : \exp(-|\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger - a^\dagger a - b^\dagger b + \eta^* a - \eta b + ab) : \\ &=: \exp\left((a^\dagger - b)(a - b^\dagger) + a^\dagger b^\dagger + ab - a^\dagger a - b^\dagger b\right) := 1, \end{aligned} \quad (11)$$

where  $d^2\eta \equiv d\eta_1 d\eta_2$  and we have used an expression  $|00\rangle\langle 00| =: \exp(-a^\dagger a - b^\dagger b) :$  and the notation  $: \dots :$  stands for taking the normal product of the creation and annihilation operators. It is easily to derive the inner product of the states  $|\eta\rangle$

$$\langle\eta|\eta'\rangle = \delta^{(2)}(\eta - \eta') = \delta(\eta_1 - \eta'_1)\delta(\eta_2 - \eta'_2). \quad (12)$$

Therefore, the eigenstates  $|\eta\rangle$  form an orthonormal and complete set of base vectors and can be used to expand any other state-vector in the related Hilbert space, so these state form a representation for NCQM.

Similarly, we may express the operators  $\hat{S}$  and  $\hat{K}$  as

$$\hat{S} = \frac{1}{2}(a + a^\dagger + b + b^\dagger) + \frac{\theta}{4i}(a - a^\dagger - b + b^\dagger), \quad \hat{K} = \frac{1}{2i}(a - a^\dagger - b + b^\dagger). \quad (13)$$

The common eigenstate of  $\hat{S}$  and  $\hat{K}$  is

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|\xi|^2}{2} + \xi a^\dagger + \xi^* b^\dagger - a^\dagger b^\dagger\right) |00\rangle. \quad (14)$$

With the aid of two expressions

$$\frac{1}{2}(a + a^\dagger + b + b^\dagger)|\xi\rangle = \xi_1|\xi\rangle, \quad \frac{1}{2i}(a - a^\dagger - b + b^\dagger)|\xi\rangle = \xi_2|\xi\rangle, \quad (15)$$

we have

$$\hat{S}|\xi\rangle = \left(\xi_1 + \frac{\theta}{2}\xi_2\right)|\xi\rangle, \quad \hat{K}|\xi\rangle = \xi_2|\xi\rangle. \quad (16)$$

Also the states  $|\xi\rangle$  form an orthonormal and complete set of base vectors

$$\int_{-\infty}^{\infty} d^2\xi |\xi\rangle\langle\xi| = 1, \quad \langle\xi|\xi'\rangle = \delta^{(2)}(\xi - \xi') = \delta(\xi_1 - \xi'_1)\delta(\xi_2 - \xi'_2), \quad (17)$$

here  $d^2\xi \equiv d\xi_1 d\xi_2$ .

Thus the eigenstates  $|\eta\rangle$  and  $|\xi\rangle$  form two representations for quantum mechanics on the noncommutative plane, respectively. In the next section we will explain that in fact the states  $|\eta\rangle$

and  $|\xi\rangle$  basically are entangled states in the noncommutative plane, so we may call the  $|\eta\rangle$  and  $|\xi\rangle$  representations as entangled state representations. For the noncommutative quantum plane, sometimes working in the  $|\eta\rangle$  or  $|\xi\rangle$  representation is more convenient, so we first need to know the scalar product of  $|\eta\rangle$  and  $|\xi\rangle$ . With the aid of over-completeness of coherent states

$$\int \frac{d^2 z_1 d^2 z_2}{\pi^2} |z_1, z_2\rangle \langle z_1, z_2| = 1, \quad (18)$$

where  $|z_1, z_2\rangle$  is a two-mode canonical coherent state

$$|z_1, z_2\rangle = |z_1\rangle_a |z_2\rangle_b = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} e^{z_1 a^\dagger + z_2 b^\dagger} |00\rangle, \quad (19)$$

one may simply get

$$\langle \eta | \xi \rangle = \int \frac{d^2 z_1 d^2 z_2}{\pi^2} \langle \eta | z_1, z_2 \rangle \langle z_1, z_2 | \xi \rangle = \frac{1}{2\pi} e^{i(\eta_1 \xi_2 - \eta_2 \xi_1)}. \quad (20)$$

Having the eq.(20), one easily gets all of matrix elements of the basic operators  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{P}_x$  and  $\hat{P}_y$  on the noncommutative plane in the entangled state representation  $|\eta\rangle$ . To do this, we only need to evaluate  $\langle \eta | \hat{S} | \eta' \rangle$  and  $\langle \eta | \hat{K} | \eta' \rangle$ , and obtain

$$\langle \eta | \hat{S} | \eta' \rangle = \langle \eta | \hat{S} \int d^2 \xi | \xi \rangle \langle \xi | \eta' \rangle = i \left( \frac{\partial}{\partial \eta_2} - \frac{\theta}{2} \frac{\partial}{\partial \eta_1} \right) \delta^{(2)}(\eta - \eta'), \quad (21)$$

and

$$\langle \eta | \hat{K} | \eta' \rangle = \langle \eta | \hat{K} \int d^2 \xi | \xi \rangle \langle \xi | \eta' \rangle = -i \frac{\partial}{\partial \eta_1} \delta^{(2)}(\eta - \eta'). \quad (22)$$

Thus in the  $|\eta\rangle$  representation, we have

$$\begin{aligned} \langle \eta | \hat{X} | \eta' \rangle &= \frac{1}{\sqrt{2}} \left( \eta_1 + i \partial_{\eta_2} - \frac{\theta}{2} (\eta_2 + i \partial_{\eta_1}) \right) \delta^{(2)}(\eta - \eta'), \\ \langle \eta | \hat{Y} | \eta' \rangle &= \frac{-1}{\sqrt{2}} \left( \eta_1 - i \partial_{\eta_2} - \frac{\theta}{2} (\eta_2 - i \partial_{\eta_1}) \right) \delta^{(2)}(\eta - \eta'), \\ \langle \eta | \hat{P}_x | \eta' \rangle &= \frac{1}{\sqrt{2}} (\eta_2 - i \partial_{\eta_1}) \delta^{(2)}(\eta - \eta'), \\ \langle \eta | \hat{P}_y | \eta' \rangle &= \frac{1}{\sqrt{2}} (\eta_2 + i \partial_{\eta_1}) \delta^{(2)}(\eta - \eta'). \end{aligned} \quad (23)$$

Similarly, in the  $|\xi\rangle$  representation, we have

$$\begin{aligned} \langle \xi | \hat{X} | \xi' \rangle &= \frac{1}{\sqrt{2}} \left( \xi_1 + i \partial_{\xi_2} + \frac{\theta}{2} (\xi_2 + i \partial_{\xi_1}) \right) \delta^{(2)}(\xi - \xi'), \\ \langle \xi | \hat{Y} | \xi' \rangle &= \frac{1}{\sqrt{2}} \left( \xi_1 - i \partial_{\xi_2} + \frac{\theta}{2} (\xi_2 - i \partial_{\xi_1}) \right) \delta^{(2)}(\xi - \xi'), \\ \langle \xi | \hat{P}_x | \xi' \rangle &= \frac{1}{\sqrt{2}} (\xi_2 - i \partial_{\xi_1}) \delta^{(2)}(\xi - \xi'), \\ \langle \xi | \hat{P}_y | \xi' \rangle &= \frac{-1}{\sqrt{2}} (\xi_2 + i \partial_{\xi_1}) \delta^{(2)}(\xi - \xi'). \end{aligned} \quad (24)$$

### 3 Entanglement properties of the states $|\eta\rangle$ and $|\xi\rangle$

From eq.(8) and eq.(14) we find that there exists intrinsic entanglement of different degrees of freedom corresponding to different coordinate components on a noncommutative plane. Usually, these states are so-called entangled states, therefore we may name these two representations as entangled state representations. In order to show this kind of entanglement more explicitly, let us consider Fourier transform of the state  $|\eta\rangle$ . Using a familiar expression for eigenstate  $|q\rangle$  of coordinate operator  $x$  ( $x|q\rangle = q|q\rangle$ ) in Fock space

$$|q\rangle_a = \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \quad (25)$$

one can write the Fourier transform of  $|\eta\rangle$  as

$$\int_{-\infty}^{\infty} \frac{d\eta_2}{\sqrt{2\pi}} |\eta\rangle e^{-i\eta_2} = \left| \frac{u + \eta_1}{\sqrt{2}} \right\rangle_a \left| \frac{u - \eta_1}{\sqrt{2}} \right\rangle_b. \quad (26)$$

If one furthermore consider inverse Fourier transform of the above expression, one will get

$$|\eta\rangle = \frac{1}{\sqrt{\pi}} e^{-i\eta_1\eta_2} \int_{-\infty}^{\infty} dq \left| q \right\rangle_a \left| q - \sqrt{2}\eta_1 \right\rangle_b e^{i\sqrt{2}\eta_2q}. \quad (27)$$

This is exactly the well-known Schmidt decomposition of a pure state which expresses the pure state can not be factorized as a direct product of two other states and therefore is an entangled state. On the other hand, noticing expression for eigenstate  $|p\rangle$  of momentum operator  $p$  in the Fock space

$$|p\rangle_a = \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{p^2}{2} + i\sqrt{2}pa^\dagger + \frac{a^{\dagger 2}}{2}\right)|0\rangle, \quad (28)$$

one can also derive

$$\int_{-\infty}^{\infty} \frac{d\eta_1}{\sqrt{2\pi}} |\eta\rangle e^{i\eta_1} = \left| \frac{v + \eta_2}{\sqrt{2}} \right\rangle_a \left| \frac{-v + \eta_2}{\sqrt{2}} \right\rangle_b \quad (29)$$

in terms of the eigenstates of the momentum operator whose inverse Fourier transform leads to another standard expression for an entangled state

$$|\eta\rangle = \frac{1}{\sqrt{\pi}} e^{-i\eta_1\eta_2} \int_{-\infty}^{\infty} dp \left| p + \sqrt{2}\eta_2 \right\rangle_a \left| -p \right\rangle_b e^{-i\sqrt{2}\eta_1p}. \quad (30)$$

For the eigenstate  $|\xi\rangle$ , using the eigenstate  $|q\rangle$  of coordinate operator (eq.(25)), one has similarly

$$\int_{-\infty}^{\infty} \frac{d\xi_2}{\sqrt{2\pi}} |\xi\rangle e^{-i\xi_2} = \left| \frac{u + \xi_1}{\sqrt{2}} \right\rangle_a \left| \frac{u - \xi_1}{\sqrt{2}} \right\rangle_b. \quad (31)$$

Its inverse Fourier transform is the Schmidt decomposition of the state  $|\xi\rangle$

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} e^{i\xi_1\xi_2} \int_{-\infty}^{\infty} dq \left| q + \sqrt{2}\xi_1 \right\rangle_a \left| -q \right\rangle_b e^{i\sqrt{2}\xi_2q}. \quad (32)$$

Of course, in terms of the eigenstate  $|p\rangle$  of the momentum operator (eq.(28)), one can get

$$\int_{-\infty}^{\infty} \frac{d\xi_1}{\sqrt{2\pi}} |\xi\rangle e^{iv\xi_1} = \left| \frac{v+\xi_2}{\sqrt{2}} \right\rangle_a \left| \frac{v-\xi_2}{\sqrt{2}} \right\rangle_b, \quad (33)$$

and its inverse transform leads to another Schmidt decomposition of the state  $|\xi\rangle$

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} e^{i\xi_1\xi_2} \int_{-\infty}^{\infty} dp \left| p \right\rangle_a \left| p - \sqrt{2}\xi_2 \right\rangle_b e^{-i\sqrt{2}\xi_1 p}. \quad (34)$$

Therefore it is reasonable to call the  $|\eta\rangle$  and  $|\xi\rangle$  representations as entangled state representations.

## 4 Unitary transformations

In the past years NCQM was discussed extensively in various aspect. The most popular method of formulating NCQM in the vast literature is treating the coordinates as commuting, but introducing  $*_{\theta}$ -product (for instance, in the noncommutative plane,  $*_{\theta} \equiv \exp \frac{i\theta}{2} \left( \overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x \right)$ ) between functions on the noncommutative space to reflect the non-commutativity of coordinates. Using the  $*_{\theta}$ -product, Schrödinger equation

$$\hat{H}(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y) |\psi\rangle = E |\psi\rangle \quad (35)$$

on the noncommutative plane should be written as [7]

$$\hat{H}(x, y, p_x, p_y) *_\theta \psi(x, y) = E \psi(x, y) \quad (36)$$

where  $\psi(x, y) = \langle x, y | \psi \rangle$ , the operators  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{P}_x$  and  $\hat{P}_y$  satisfy the commutation relations (1), and the operators  $x$ ,  $y$ ,  $p_x$  and  $p_y$  satisfy the commutation relations (4), respectively. In fact, in eq.(36) people have used the representation  $|x, y\rangle = |x\rangle_a |y\rangle_b$  which is common eigenstate of the operators  $x$  and  $y$  (not the operators  $\hat{X}$  and  $\hat{Y}$ ), so we would like to name it the "quasi-coordinate" representation. In the  $|x, y\rangle$  representation one can simply write out the matrix elements of the operators  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{P}_x$  and  $\hat{P}_y$  on the noncommutative plane

$$\begin{aligned} \langle x, y | \hat{X} | x', y' \rangle &= \left( x + \frac{i\theta}{2} \partial_y \right) \delta(x - x') \delta(y - y'), \\ \langle x, y | \hat{Y} | x', y' \rangle &= \left( y - \frac{i\theta}{2} \partial_x \right) \delta(x - x') \delta(y - y'), \\ \langle x, y | \hat{P}_x | x', y' \rangle &= -i \partial_x \delta(x - x') \delta(y - y'), \\ \langle x, y | \hat{P}_y | x', y' \rangle &= -i \partial_y \delta(x - x') \delta(y - y'). \end{aligned} \quad (37)$$

From eq.(3) we know that the states  $|x, y\rangle$  are also common eigenstates of the operators  $\hat{X} + \frac{\theta}{2} \hat{P}_y$  and  $\hat{Y} - \frac{\theta}{2} \hat{P}_x$ , so there is a unitary transformation between the two representations ( $|\eta\rangle$  and  $|x, y\rangle$ ) whose matrix elements may be written as

$$\langle \eta | x, y \rangle = \frac{1}{\sqrt{\pi}} e^{i(\eta_1 - \sqrt{2}x)\eta_2} \delta(x - y - \sqrt{2}\eta_1), \quad (38)$$

where eq.(27) is used. Similarly, using eq.(32), one can get matrix elements of the transformation between another two representations ( $|\xi\rangle$  and  $|x, y\rangle$ )

$$\langle \xi | x, y \rangle = \frac{1}{\sqrt{\pi}} e^{-i(\xi_1 - \sqrt{2}y)\xi_2} \delta(x + y - \sqrt{2}\xi_1). \quad (39)$$

Using the completeness relations eqs.(11), (17) and  $\int dx dy |x, y\rangle \langle x, y| = 1$ , one can easily see that eqs.(38) and (39) indeed present the unitary transformations between the entangled state representations and the "quasi-coordinate" representation  $|x, y\rangle$ .

In order to get an clear form of the unitary transformation between the  $|\eta\rangle$  and the  $|x, y\rangle$  representations, let us consider the following integration built from the entangled state  $|\eta\rangle$  and the two-mode "quasi-coordinate" eigenstate  $|x, y\rangle$

$$\begin{aligned} U &= \int d^2\eta |x, y\rangle \langle \eta| \Big|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}} \\ &= \int \frac{d\eta_1 d\eta_2}{\pi} \exp\left(-\frac{x^2}{2} - \frac{y^2}{2} + \sqrt{2}x a^\dagger + \sqrt{2}y b^\dagger - \frac{a^{\dagger 2}}{2} - \frac{b^{\dagger 2}}{2}\right) \\ &\quad |00\rangle \langle 00| \exp\left(-\frac{|\eta|^2}{2} + \eta^* a - \eta b + a b\right) \Big|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}}, \end{aligned} \quad (40)$$

here we have taken all parameters in the ordinary harmonic oscillator expressions (i.e.  $m, \omega$ ) equal to 1 for the simplicity. Using the trick in the calculation of eq.(11), we obtain

$$U = : \exp\left(-\frac{1+i}{2}(a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a)\right) :, \quad (41)$$

where the result of the integration is expressed in terms of the normal ordered product. To show the unitary property of the operator  $U$  more clearly, introducing an operator  $e^{irS}$  with  $S = a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a$ , we have  $[S, a^\dagger] = a^\dagger + b^\dagger$  and  $[S, b^\dagger] = a^\dagger + b^\dagger$ , which lead to

$$e^{irS} a^\dagger e^{-irS} = \frac{e^{2ir} + 1}{2} a^\dagger + \frac{e^{2ir} - 1}{2} b^\dagger, \quad e^{irS} b^\dagger e^{-irS} = \frac{e^{2ir} - 1}{2} a^\dagger + \frac{e^{2ir} + 1}{2} b^\dagger, \quad (42)$$

and further

$$\begin{aligned} e^{irS} &= e^{irS} \sum_{n,m=0}^{\infty} |n, m\rangle \langle n, m| \\ &= e^{irS} \sum_{n,m=0}^{\infty} \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n! m!}} |00\rangle \langle 00| \frac{a^n b^m}{\sqrt{n! m!}} = : \exp\left(-\frac{1 - e^{2ir}}{2} S\right) :. \end{aligned} \quad (43)$$

When choosing  $r = -\pi/4$ , we have

$$U = e^{-\frac{i\pi}{4}(a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a)}, \quad (44)$$

which is unitary obviously. From eq.(44), it is easily to get

$$Ua^\dagger U^\dagger = \frac{1-i}{2}a^\dagger - \frac{1+i}{2}b^\dagger, \quad Ub^\dagger U^\dagger = -\frac{1+i}{2}a^\dagger + \frac{1-i}{2}b^\dagger, \quad (45)$$

which lead to

$$U|\eta\rangle = |x, y\rangle \Big|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}}, \quad U|x, y\rangle = |\eta^*\rangle \Big|_{\eta_1=\frac{x-y}{\sqrt{2}}, \eta_2=\frac{x+y}{\sqrt{2}}}. \quad (46)$$

Thus  $U$  indeed transfer the state  $|\eta\rangle$  to the state  $|x, y\rangle$  and vice versa. In the  $|\eta\rangle$  representation one can calculate the following matrix element of the operator  $U$ :  $\langle \eta|U|\zeta\rangle$ , where  $\zeta = \zeta_1 + i\zeta_2$ . Using eq.(46) one has

$$\langle \eta|U|\zeta\rangle = \langle \eta|x, y\rangle \Big|_{x=\frac{\zeta_1+\zeta_2}{\sqrt{2}}, y=\frac{\zeta_2-\zeta_1}{\sqrt{2}}} = \frac{1}{\sqrt{\pi}} e^{i(\eta_1-\zeta_1-\zeta_2)\eta_2} \delta(\sqrt{2}\zeta_1 - \sqrt{2}\eta_1). \quad (47)$$

If one further takes  $\zeta_1 = (x-y)/\sqrt{2}$  and  $\zeta_2 = (x+y)/\sqrt{2}$ , eq.(47) will lead to eq.(38) exactly. This means that eq.(38) is just a matrix element of the unitary operator  $U$  in the entangled state representation. Similarly one can find out the unitary transformation between the  $|\xi\rangle$  and the  $|x, y\rangle$  representations and endow eq.(39) with the same explanation.

Having the unitary transformation (38), and using

$$\langle \eta|\hat{F}|\eta'\rangle = \int dx dy dx' dy' \langle \eta|x, y\rangle \langle x, y|\hat{F}|x', y'\rangle \langle x', y'|\eta'\rangle, \quad (48)$$

or

$$\langle x, y|\hat{F}|x', y'\rangle = \int d^2\eta d^2\eta' \langle x, y|\eta\rangle \langle \eta|\hat{F}|\eta'\rangle \langle \eta'|x', y'\rangle, \quad (49)$$

one may get the matrix elements of any operator  $\hat{F}$  in one representation, if one knows  $\hat{F}$  in another representation. For example, taking  $\hat{F} = \hat{P}_x$ , one has

$$\begin{aligned} \langle \eta|\hat{P}_x|\eta'\rangle &= \int \frac{dx dy dx' dy'}{\pi} e^{i(\eta_1-\sqrt{2}x)\eta_2} \delta(x-y-\sqrt{2}\eta_1) (-i\partial_x) \delta(x-x') \delta(y-y') \\ &\quad e^{-i(\eta'_1-\sqrt{2}x')\eta'_2} \delta(x'-y'-\sqrt{2}\eta'_1) \\ &= e^{i\eta_1(\eta_2-\eta'_2)} \left( \sqrt{2}\eta_2 - i\frac{\partial}{\partial\sqrt{2}\eta_1} \right) \delta(\eta_1-\eta'_1) \delta(\eta_2-\eta'_2) \\ &= \frac{1}{\sqrt{2}} (\eta_2 - i\partial_{\eta_1}) \delta^2(\eta - \eta') \end{aligned} \quad (50)$$

which exactly coincides with eq.(23). Similarly, one also has

$$\begin{aligned} \langle \eta|\hat{P}_y|\eta'\rangle &= e^{i\eta_1(\eta_2-\eta'_2)} \left( i\frac{\partial}{\partial\sqrt{2}\eta_1} \right) \delta(\eta_1-\eta'_1) \delta(\eta_2-\eta'_2) \\ &= \frac{1}{\sqrt{2}} (\eta_2 + i\partial_{\eta_1}) \delta^2(\eta - \eta'). \end{aligned} \quad (51)$$

Noticing

$$x e^{i(\eta_1 - \sqrt{2}x)\eta_2} = \frac{1}{\sqrt{2}} (\eta_1 + i \partial_{\eta_2}) e^{i(\eta_1 - \sqrt{2}x)\eta_2}, \quad (52)$$

one can obtain other two expressions of eq.(23). Of course, with the aid of the unitary transformation (39), from eq.(37) one may get eq.(24).

Therefore, we derive the unitary transformations which change the  $|x, y\rangle$  representation to the  $|\eta\rangle$  (or  $|\xi\rangle$ ) representation and vice versa.

## 5 Eigenfunctions of some basic operators in the entangled state representations

From the section 2 we see that wave function of any state vector  $|\psi\rangle$  in the entangled state representation  $|\eta\rangle$  can be expressed as  $\psi(\eta) = \psi(\eta_1, \eta_2) = \langle \eta | \psi \rangle$ . Eq.(23) gives the representations of the operators  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{P}_x$  and  $\hat{P}_y$  in the entangled state representation  $|\eta\rangle$ , for example, the operators  $\hat{P}_x$  and  $\hat{P}_y$  can be replaced by  $\frac{1}{\sqrt{2}}(\eta_2 - i\partial_{\eta_1})$  and  $\frac{1}{\sqrt{2}}(\eta_2 + i\partial_{\eta_1})$  respectively. It is well known that eigenfunctions of the operators  $\hat{P}_x$  and  $\hat{P}_y$  in ordinary quantum mechanics may be the plane waves, so it is interesting to see what are the eigenfunctions of the same operators in NCQM. Since the operators  $\hat{P}_x$  and  $\hat{P}_y$  are commuting each other and have common eigenstates, we first derive their common eigenfunctions in the entangled state representation.

Noticing eqs.(10) and (22), and  $[\hat{P}, \hat{K}] = 0$ , and denoting common eigenstate of  $\hat{P}$  and  $\hat{K}$  as  $|\psi\rangle$  with eigenvalues  $p$  and  $k$  respectively, in the entangled state representation we have

$$\psi(\eta) = \frac{1}{\sqrt{2\pi}} \delta(\eta_2 - p) e^{ik\eta_1}, \quad (53)$$

where  $\frac{1}{\sqrt{2\pi}}$  is normalization constant. Because the operators  $\hat{P}_x$  and  $\hat{P}_y$  are linear combination of  $\hat{P}$  and  $\hat{K}$ , if we denote the eigenstate of  $\hat{P}_x$  and  $\hat{P}_y$  as  $|\psi_{p_x, p_y}\rangle$  with eigenvalues  $p_x = \frac{1}{\sqrt{2}}(p + k)$  and  $p_y = \frac{1}{\sqrt{2}}(p - k)$  respectively, we have the common eigenfunction of  $\hat{P}_x$  and  $\hat{P}_y$

$$\psi_{p_x, p_y}(\eta) = \frac{1}{\sqrt{2\pi}} \delta\left(\eta_2 - \frac{p_x + p_y}{\sqrt{2}}\right) e^{i(p_x - p_y)\eta_1 / \sqrt{2}}. \quad (54)$$

It is clearly shown that the eigenfunctions of  $\hat{P}_x$  and  $\hat{P}_y$  in  $|\eta\rangle$  representation are entanglement of ordinary coordinate and momentum eigenfunctions. On the other hand, in the  $|x, y\rangle$  representation, the eigenfunctions of  $\hat{P}_x$  and  $\hat{P}_y$  are simply

$$\psi_{p_x, p_y}(x, y) = \langle x, y | \psi_{p_x, p_y} \rangle = \frac{1}{2\pi} e^{i(p_x x + p_y y)}. \quad (55)$$

Using the representation transformation (38), one has

$$\begin{aligned} \psi_{p_x, p_y}(\eta) &= \int dx dy \langle \eta | x, y \rangle \langle x, y | \psi_{p_x, p_y} \rangle \\ &= \int \frac{dx dy}{2\pi^{3/2}} \delta(x - y - \sqrt{2}\eta_1) e^{i((\eta_1 - \sqrt{2}x)\eta_2 + p_x x + p_y y)} \end{aligned} \quad (56)$$

which leads to the right hand side of eq.(54) exactly. It also means that if one wants to find eigenfunction of some operator in the  $|\eta\rangle$  representation, one can first get the eigenfunction in the  $|x, y\rangle$  representation and then derive the eigenfunction in the  $|\eta\rangle$  representation by the representation transformation, and vice versa.

Now let us use this method to find out eigenfunction of  $\hat{X}$  in the entangled state representation  $|\eta\rangle$ . In fact it is easily understood that the eigenfunction of  $\hat{X}$  is degenerate and in order to remove the degeneracy we should consider common eigenfunctions of the operators  $\hat{X}$  and  $\hat{P}_y$ . From eq.(3) we see that the normalized eigenfunction of  $\hat{X}$  and  $\hat{P}_y$  in the  $|x, y\rangle$  representation can be expressed as

$$\psi_{X,p_y}(x, y) = \frac{1}{\sqrt{2\pi}} \delta(x - \tilde{x}) e^{i\tilde{p}_y y}, \quad (57)$$

where  $\tilde{p}_y$  is the eigenvalue of the operator  $\hat{P}_y$  and  $\tilde{x} - \frac{\theta}{2}\tilde{p}_y \equiv \tilde{X}$  the eigenvalue of  $\hat{X}$  respectively. Then using the transformation (38) we get the common eigenfunction of  $\hat{X}$  and  $\hat{P}_y$  in the  $|\eta\rangle$  representation

$$\psi_{X,p_y}(\eta) = \frac{1}{\sqrt{2}\pi} e^{i\eta_1 \eta_2} e^{-i\sqrt{2}(\tilde{X}\eta_2 + \tilde{p}_y(\eta_1 + \frac{\theta}{2}\eta_2))} e^{i\tilde{p}_y(\tilde{X} + \frac{\theta}{2}\tilde{p}_y)}. \quad (58)$$

Furthermore noticing that in the  $|\eta\rangle$  representation  $\hat{X}$  and  $\hat{P}_y$  can be expressed as

$$\hat{X} = \frac{1}{\sqrt{2}} \left( \eta_1 + i\partial_{\eta_2} - \frac{\theta}{2}\eta_2 - i\frac{\theta}{2}\partial_{\eta_1} \right) \quad (59)$$

and

$$\hat{P}_y = \frac{1}{\sqrt{2}} (\eta_2 + \partial_{\eta_1}) \quad (60)$$

respectively, it is straightforwardly to check  $\hat{X}\psi_{X,p_y}(\eta) = \tilde{X}\psi_{X,p_y}(\eta)$  and  $\hat{P}_y\psi_{X,p_y}(\eta) = \tilde{p}_y\psi_{X,p_y}(\eta)$ . Of course, one can also directly use the expression (59) and solves differential equation  $\hat{X}\psi(\eta) = \tilde{X}\psi(\eta)$  in the  $|\eta\rangle$  representation to get the eigenfunction  $\psi(\eta)$  of  $\hat{X}$ .

Similarly we have common eigenfunction of  $\hat{Y}$  and  $\hat{P}_x$  in the  $|\eta\rangle$  representation

$$\psi_{Y,p_x}(\eta) = \frac{1}{\sqrt{2}\pi} e^{-i\eta_1 \eta_2} e^{-i\sqrt{2}(\tilde{Y}\eta_2 - \tilde{p}_x(\eta_1 + \frac{\theta}{2}\eta_2))} e^{i\tilde{p}_x(\tilde{Y} - \frac{\theta}{2}\tilde{p}_x)}, \quad (61)$$

where  $\tilde{p}_x$  and  $\tilde{y} + \frac{\theta}{2}\tilde{p}_x \equiv \tilde{Y}$  are the eigenvalue of  $\hat{P}_x$  and  $\hat{Y}$  respectively.

From eqs.(58) and (61) we see that the eigenfunctions of the operators  $\hat{X}$  and  $\hat{Y}$  in the entangled state representation are not simply plane waves. In fact, they also display some kind of entanglement of the coordinates and the momenta. Of course, one can also obtain eigenfunctions of these operators in the  $|\xi\rangle$  representation.

## 6 Some possible applications

It is well know that representation plays a basic role in quantum mechanics like the coordinate systems in geometry. In section 2 we introduced the entangled state representations  $|\eta\rangle$  and  $|\xi\rangle$ ,

which are related to the  $|x, y\rangle$  representation by unitary transformations as shown in section 4. In the  $|\eta\rangle$  or  $|\xi\rangle$  representation one can also solve Schrödinger equation of NCQM as in the  $|x, y\rangle$  representation, and sometimes it is more convenient working in the entangled state representation than in the  $|x, y\rangle$  representation. To show this, let us study a two-dimensional harmonic oscillator on the noncommutative plane with both momentum-momentum (kinetic) coupling and coordinate-coordinate (elastic) coupling. The Hamiltonian is

$$H = \frac{1}{2}\hat{P}_x^2 + \frac{1}{2}\hat{P}_y^2 + \frac{1}{2}\hat{X}^2 + \frac{1}{2}\hat{Y}^2 + \kappa\hat{P}_x\hat{P}_y + \frac{\lambda}{2}(\hat{X}\hat{Y} + \hat{Y}\hat{X}), \quad (62)$$

where the operators  $\hat{P}_x$ ,  $\hat{P}_y$ ,  $\hat{X}$  and  $\hat{Y}$  satisfy the commutation relations (1). After substituting eq.(3) into eq.(62) we get the Hamiltonian  $H$  in the  $|x, y\rangle$  representation

$$\begin{aligned} H = & \frac{1}{2}\left(1 + \frac{\theta}{4}\right)p_x^2 + \frac{1}{2}\left(1 + \frac{\theta}{4}\right)p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 \\ & + \left(\kappa - \frac{\lambda\theta^2}{4}\right)p_xp_y + \lambda xy - \frac{\theta}{2}(xP_y - yp_x) + \frac{\lambda\theta}{2}(xp_x - yp_y), \end{aligned} \quad (63)$$

which includes not only the kinetic and the elastic coupling terms, but also the coordinate-momentum coupling terms (they are the angular momentum term and the squeezing term, respectively). It is not an easy task to solve its eigenequation. However, in the  $|\eta\rangle$  representation the Hamiltonian  $H$  has simpler form

$$\begin{aligned} H = & \frac{1}{2}\left(1 + \frac{\theta^2}{2} - \kappa + \frac{\lambda\theta^2}{4}\right)p_1^2 + \frac{1}{2}(1 + \lambda)p_2^2 - \frac{\theta}{2}(1 + \lambda)p_1p_2 \\ & + \frac{1}{2}(1 - \lambda)\eta_1^2 + \frac{1}{2}\left(1 + \frac{\theta^2}{2} + \kappa - \frac{\lambda\theta^2}{4}\right)\eta_2^2 - \frac{\theta}{2}(1 - \lambda)\eta_1\eta_2, \end{aligned} \quad (64)$$

where  $p_i = -i\partial/\partial\eta_i$  ( $i=1,2$ ). In the Hamiltonian (64), only the kinetic and the elastic coupling terms survive, and it is easier to be handled than the form (63). Of course, it is needless to emphasize that the Hamiltonian (63) and (64) are connected via a unitary transformation described in section 4.

Before diagonalizing  $H$ , let us introduce some notations to rewrite (64) so that it has more familiar form

$$\begin{aligned} m_1 = & \left(1 + \frac{\theta^2}{2} - \kappa + \frac{\lambda\theta^2}{4}\right)^{-1}, \quad \omega_1 = \sqrt{(1 - \lambda)\left(1 + \frac{\theta^2}{2} - \kappa + \frac{\lambda\theta^2}{4}\right)}, \quad \alpha = \frac{\theta}{2}(1 + \lambda), \\ m_2 = & (1 + \lambda)^{-1}, \quad \omega_2 = \sqrt{(1 + \lambda)\left(1 + \frac{\theta^2}{2} + \kappa - \frac{\lambda\theta^2}{4}\right)}, \quad \beta = \frac{\theta}{2}(1 - \lambda). \end{aligned} \quad (65)$$

In terms of these notations,  $H$  becomes

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \alpha p_1p_2 + \frac{m_1\omega_1^2}{2}\eta_1^2 + \frac{m_2\omega_2^2}{2}\eta_2^2 - \beta\eta_1\eta_2. \quad (66)$$

Now let us introduce a two by two matrix  $A$  whose matrix elements  $a_{ij}$  will be determined later ( $i, j = 1, 2$ ). If we use  $\vec{p}$  to denote the two-dimensional momentum  $(p_1, p_2)$ , one can write  $\vec{p} = A\vec{p}' = (\tilde{p}_1, \tilde{p}_2)$  with  $\tilde{p}_i = a_{ij}p_j$ , and inversely,  $p_i = b_{ij}\tilde{p}_j$ , where  $b_{ij}$  are elements of inverse matrix of  $A$ . Consider the following transformation

$$V = \sqrt{\det A} \int d\vec{p} |A\vec{p}'\rangle \langle \vec{p}| \quad (67)$$

in the Hilbert space spanned by two-mode momentum eigenstates  $|\vec{p}'\rangle$ , which is unitary clearly

$$\begin{aligned} V V^\dagger &= \det A \int d\vec{p} d\vec{p}' |A\vec{p}'\rangle \langle \vec{p}'| \vec{p}' \rangle \langle A\vec{p}'| \\ &= \det A \int d\vec{p} |A\vec{p}'\rangle \langle A\vec{p}'| = \int d\vec{p} |\vec{p}'\rangle \langle \vec{p}'| = 1, \end{aligned} \quad (68)$$

and similarly  $V^\dagger V = 1$ . In eq.(67),  $|\vec{p}'\rangle = |p_1\rangle |p_2\rangle$  and  $|p_i\rangle$  are the momentum eigenstates

$$|p_i\rangle = \left( \frac{1}{\pi m_i \omega_i} \right)^{1/4} \exp \left( -\frac{p_i^2}{2m_i \omega_i} + i \sqrt{\frac{2}{m_i \omega_i}} p_i a_i^\dagger + \frac{1}{2} a_i^{\dagger 2} \right) |0\rangle_i, \quad (69)$$

where  $a_i^\dagger$  (and  $a_i$ ) are the ordinary bosonic creation (and annihilation) operators

$$a_i = \frac{1}{2} \left( \sqrt{m_i \omega_i} \eta_i + i \frac{1}{\sqrt{m_i \omega_i}} p_i \right), \quad a_i^\dagger = \frac{1}{2} \left( \sqrt{m_i \omega_i} \eta_i - i \frac{1}{\sqrt{m_i \omega_i}} p_i \right). \quad (70)$$

It is not difficult to see that  $V$  transforms  $V p_i V^\dagger = b_{ij} p_j$  and  $V \eta_i V^\dagger = a_{ji} \eta_j$ , because

$$V p_i V^\dagger = \det A \int d\vec{p} d\vec{p}' |A\vec{p}'\rangle \langle \vec{p}'| p_i |\vec{p}'\rangle \langle A\vec{p}'| = \int d\vec{p} b_{ij} \tilde{p}_j |\vec{p}'\rangle \langle \vec{p}'| = b_{ij} p_j, \quad (71)$$

and

$$V \eta_i V^\dagger = \det A \int d\vec{p} d\vec{p}' |A\vec{p}'\rangle \langle \vec{p}'| \frac{\partial}{\partial p_i} \delta(\vec{p} - \vec{p}') \langle A\vec{p}'|, \quad (72)$$

furthermore, acting eq.(72) from the right-hand side on  $\langle \vec{\eta}|$  leads to

$$\langle \vec{\eta}| V \eta_i V^\dagger = \det A \int d\vec{p} d\vec{p}' \left( -i \frac{\partial}{\partial p_i} \exp(i a_{jk} p_k \eta_j) \right) \delta(\vec{p} - \vec{p}') \langle A\vec{p}'| = \langle \vec{\eta}| a_{ji} \eta_j, \quad (73)$$

which means that  $V \eta_i V^\dagger = a_{ji} \eta_j$ .

Now let us act the unitary transformation  $V$  on the Hamiltonian (66) and get

$$\begin{aligned} V H V^\dagger &= \frac{1}{2m_1} (b_{11} p_1 + b_{12} p_2)^2 + \frac{1}{2m_2} (b_{21} p_1 + b_{22} p_2)^2 - \alpha (b_{11} p_1 + b_{12} p_2) (b_{21} p_1 + b_{22} p_2) \\ &\quad + \frac{m_1 \omega_1^2}{2} (a_{11} \eta_1 + a_{21} \eta_2)^2 + \frac{m_2 \omega_2^2}{2} (a_{12} \eta_1 + a_{22} \eta_2)^2 \\ &\quad - \beta (a_{11} \eta_1 + a_{21} \eta_2) (a_{12} \eta_1 + a_{22} \eta_2). \end{aligned} \quad (74)$$

Then in order to annihilate the coupling terms in eq.(74), we set

$$\begin{aligned}\frac{1}{m_1}a_{22}a_{12} + \frac{1}{m_2}a_{21}a_{22} + \alpha(a_{11}a_{22} + a_{12}a_{21}) &= 0, \\ m_1\omega_1^2 a_{11}a_{21} + m_2\omega_2^2 a_{12}a_{22} - \beta(a_{11}a_{22} + a_{12}a_{21}) &= 0.\end{aligned}\quad (75)$$

From eq.(75) we have

$$a_{12} = \frac{\Omega m_1}{2\alpha(\beta + \alpha m_1 m_2 \omega_2^2)} a_{11}, \quad a_{21} = -\frac{\Omega m_2}{2\alpha(\beta + \alpha m_1 m_2 \omega_1^2)} a_{22}, \quad (76)$$

where

$$\Omega = \alpha(\omega_1^2 - \omega_2^2) + \sqrt{\alpha^2(\omega_1^2 - \omega_2^2)^2 + 4\alpha^2\left(\frac{\beta}{m_1} + \alpha m_2 \omega_2^2\right)\left(\frac{\beta}{m_2} + \alpha m_1 \omega_1^2\right)}. \quad (77)$$

Thus eq.(74) can be written as

$$\begin{aligned}H_d &= \frac{a_{22}^2}{2m_1(\det A)^2} \left(1 + \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_1^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_1^2)^2}\right) p_1^2 \\ &+ \frac{a_{11}^2}{2m_2(\det A)^2} \left(1 - \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_2^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_2^2)^2}\right) p_2^2 \\ &+ \frac{a_{11}^2 m_1 \omega_1^2}{2} \left(1 - \frac{\beta \Omega}{\alpha \omega_1^2(\beta + \alpha m_1 m_2 \omega_2^2)} + \frac{m_1 m_2 \omega_2^2 \Omega^2}{4\alpha^2 \omega_1^2(\beta + \alpha m_1 m_2 \omega_2^2)^2}\right) \eta_1^2 \\ &+ \frac{a_{22}^2 m_2 \omega_2^2}{2} \left(1 + \frac{\beta \Omega}{\alpha \omega_2^2(\beta + \alpha m_1 m_2 \omega_1^2)} + \frac{m_1 m_2 \omega_1^2 \Omega^2}{4\alpha^2 \omega_2^2(\beta + \alpha m_1 m_2 \omega_1^2)^2}\right) \eta_2^2.\end{aligned}\quad (78)$$

Since

$$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} \left(1 + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_1^2)(\beta + \alpha m_1 m_2 \omega_2^2)}\right), \quad (79)$$

if we use the following notations

$$\begin{aligned}T_1 &= 1 + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_1^2)(\beta + \alpha m_1 m_2 \omega_2^2)}, \\ T_2 &= 1 + \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_1^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_1^2)^2}, \\ T_3 &= 1 - \frac{\beta \Omega}{\alpha \omega_1^2(\beta + \alpha m_1 m_2 \omega_2^2)} + \frac{m_1 m_2 \omega_2^2 \Omega^2}{4\alpha^2 \omega_1^2(\beta + \alpha m_1 m_2 \omega_2^2)^2}, \\ T_4 &= 1 - \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_2^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2(\beta + \alpha m_1 m_2 \omega_2^2)^2}, \\ T_5 &= 1 + \frac{\beta \Omega}{\alpha \omega_2^2(\beta + \alpha m_1 m_2 \omega_1^2)} + \frac{m_1 m_2 \omega_1^2 \Omega^2}{4\alpha^2 \omega_2^2(\beta + \alpha m_1 m_2 \omega_1^2)^2},\end{aligned}\quad (80)$$

and in eq.(78), let the coefficients of the terms  $p_1^2/2m_1$  and  $p_2^2/2m_2$  be equal to the coefficients of the terms  $m_1\omega_1^2\eta_1^2/2$  and  $m_2\omega_2^2\eta_2^2/2$  respectively, and denote them as  $\Lambda_1$  and  $\Lambda_2$ , we have

$$\begin{aligned} a_{11} &= T_1^{-1/2}T_2^{1/4}T_3^{-1/4}, & a_{22} &= T_1^{-1/2}T_4^{1/4}T_5^{-1/4}, \\ \Lambda_1 &= T_1^{-1}T_2^{1/2}T_3^{1/2}, & \Lambda_2 &= T_1^{-1}T_4^{1/2}T_5^{1/2}. \end{aligned} \quad (81)$$

Thus we diagonalize the Hamiltonian (64) and obtain

$$H_d = VHV^\dagger = \Lambda_1\omega_1 \left( a_1^\dagger a_1 + \frac{1}{2} \right) + \Lambda_2\omega_2 \left( a_2^\dagger a_2 + \frac{1}{2} \right), \quad (82)$$

which gives the energy spectrum of the two-dimensional harmonic oscillator (62) on the noncommutative plane with both the kinetic and the elastic couplings

$$E_{n,m} = \Lambda_1\omega_1 \left( n + \frac{1}{2} \right) + \Lambda_2\omega_2 \left( m + \frac{1}{2} \right). \quad (83)$$

This result, to our knowledge, has not been reported in the literature so far. In some special case, however, it reduces to well-known relevant results. For example, when the coupling constants  $\kappa$  and  $\lambda$  both vanish, the Hamiltonian (62) describes a two-dimensional harmonic oscillator without any coupling on the noncommutative plane. Eq.(83) reduces to

$$E_{n,m} = \sqrt{1 + \frac{\theta^2}{4}} (n + m + 1), \quad (84)$$

which was derived by many authors in other methods. For instance, eq.(84) coincides with [8].

## 7 Summary and discussion

In order to develop representation and transformation theory so that one can solve more dynamic problems for NCQM, in this work we introduce new representations on the noncommutative space which may be named the entangled state representations, because the state-vectors of these representations are common eigenstates of the difference (or the sum) of two different coordinate-component operators and the sum (or the difference) of two relevant momentum operators, and display some entanglements of different components on the noncommutative space. Since these state-vectors are orthonormal and satisfy the completeness relation, they form representations to formulate the NCQM. In this work we find out explicit unitary operator which can transfers the entangled state representation  $|\eta\rangle$  into the so-called "quasi-coordinate" representation  $|x, y\rangle$  used in the papers on NCQM. Similar unitary operator between the  $|\xi\rangle$  representation and the  $|x, y\rangle$  representation can be got also. To show the potential applications of new entangled representations, we solve exactly a two-dimensional harmonic oscillator with both the kinetic and the elastic couplings on the noncommutative plane. This example shows that some dynamic problems of NCQM may be easier solved in the entangled state representations.

It is also interesting to generalize the entangled state representations to describe two particles moving on the noncommutative space. Work on this direction will be presented in a separate paper.

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